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BETA PENTAGON RELATIONS

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The (quantum) pentagon relation underlies the existing constructions of three-dimensional quantum topology in the combinatorial framework of triangulations. We discuss a special class of integral pentagon relations and their relations to the Faddeev-type operator pentagon relations.

Keywords: pentagon relation, quantum dilogarithm, locally compact Abelian group

Dedicated to Professor Ludvig Faddeev on the occasion of his 80th birthday

1. Introduction

By *pentagon relation* in a broad sense, we mean any algebraic relation that can be given an interpretation in terms of Pachner's 2-3 move for triangulated three-dimensional manifolds. A particular but still large class of pentagon relations are satisfied by $6j$-symbols arising from the representation theory of Hopf algebras. The first historical example of this kind was published by Biedenharn and Elliott in 1953 [1], [2] in the framework of the quantum theory of angular momentum.

Motivated by the problem of giving an exact combinatorial formulation for partition functions in the quantum Chern–Simons theory with noncompact gauge groups, the constructions in [3], [4] are based on pentagon integral relations with a structure similar to that of the Biedenharn–Elliott relation but with the discrete variables corresponding to the equivalence classes of irreducible representations of the group $SU(2)$ replaced with continuous variables and discrete sums replaced with integrals. Here, we consider these relations in the framework of a more general unified approach and establish their relation to Faddeev-type operator pentagon relations. This unified approach is based on the theory of locally compact Abelian groups and is defined as follows.

For two sets S and T, we let S^T denote the set of all maps from T to S. For any nonnegative integer n, we let $[n]$ denote the set $\mathbb{Z}_{\geq 0} \cap \mathbb{Z}_{\leq n}$ and $\Delta[n]$ denote the standard combinatorial n-simplex seen as, for example, the power set $2^{[n]}$. We also let $\Delta[n]_i$ denote the set of i-dimensional simplexes of $\Delta[n]$.

Let A be a locally compact Abelian group together with a fixed Haar measure dx. We say that a function

$$\phi\colon [4] \times A^2 \to \mathbb{C}, \qquad (j, x, y) \mapsto \phi_j(x, y), \tag{1}$$

is of the *beta pentagon type* over A if the five-term integral relation

$$\phi_1(x, y)\phi_3(u, v) = \int_A \phi_4(uy, v\bar{z})\phi_2(xyuv\bar{z}, z)\phi_0(xv, y\bar{z})\, dz \tag{2}$$

is satisfied, where we set $\bar{x} \equiv x^{-1}$. Relation (2) itself is called the *beta pentagon relation* over A. Our motivation for it comes from the following combinatorial interpretation.

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Given a map as in (1), we define another map

$$W\colon \Delta[4]_3 \times A^{\Delta[3]_1} \to \mathbb{C} \tag{3}$$

by assigning

$$W(\partial_i\Delta[4], x) = \phi_i(x_{01}x_{23}\bar{x}_{03}\bar{x}_{12}, x_{03}x_{12}\bar{x}_{02}\bar{x}_{13}), \tag{4}$$

where x_{jk} is the x-image of the edge $\{j,k\}$. The equality

$$\prod_{i\in\{1,3\}} W(\partial_i\Delta[4], \varepsilon_i^* x) = \int_A dx_{13} \prod_{i\in\{0,2,4\}} W(\partial_i\Delta[4], \varepsilon_i^* x), \quad x \in A^{\Delta[4]_1}, \tag{5}$$

with the standard injections $\varepsilon_i\colon [3] \to [4]$, $i \in [4]$, defined by

$$\varepsilon_i(j) = \begin{cases} j, & j < i, \\ j+1 & \text{otherwise}, \end{cases} \tag{6}$$

is then a consequence of the beta pentagon relation for ϕ. Such a combinatorial interpretation can be used as a starting point for constructions of the topological quantum field theory (TQFT) type based on the combinatorics of triangulated pseudo-three-manifolds similarly to the Ponzano–Regge and Turaev–Viro models [3]–[6].

An interesting example of a function of the beta pentagon type over $\mathbb{R}$ and the reason why we use the term "beta" is given by the Euler beta function

$$B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}. \tag{7}$$

Namely, the solution is given explicitly by the formula [3]

$$\phi_j(x,y) = B\bigl(2\pi\mathsf{i}(x+y-\mathsf{i}0), -2\pi\mathsf{i}(y+\mathsf{i}0)\bigr). \tag{8}$$

This solution has not yet been used for topological applications, because it lacks the tetrahedral symmetries, but it would be interesting to see if the pentagon relation could have any significance, for example, for the Veneziano amplitude in string theory.

2. Some symmetries of the beta pentagon relation

One immediate symmetry of the beta pentagon relation is given by inversion of the group arguments. Namely, if $\phi_i(x,y)$ is of the beta pentagon type over A, then

$$\phi_i'(x,y) \equiv \phi_i(\bar{x}, \bar{y}) \tag{9}$$

is also of the beta pentagon type over A.

Let $\hat{A}$ be the Pontryagin dual of A. Interestingly, the beta pentagon relation is stable under the Fourier transform in the sense that if $\phi_i(x,y)$ is of the beta pentagon type over A, then

$$\hat{\phi}_i(\xi,\eta) \equiv \int_{A^2} \frac{\xi(y)}{\eta(x)} \phi_i(x,y)\, dx\, dy, \quad (\xi,\eta) \in \hat{A}^2, \tag{10}$$

is of the beta pentagon type over $\hat{A}$. This is verified as follows.

First, we define the partial Fourier transform $\check{f} \in \mathbb{C}^{\hat{A}\times A}$ for any integrable function $f \in \mathbb{C}^{A^2}$ by the formula

$$\check{f}(\xi, y) \equiv \int_A \bar{\xi}(x) f(x, y)\, dx, \quad (\xi, y) \in \hat{A} \times A. \tag{11}$$

We hence have

$$f(x, y) = \int_{\hat{A}} \xi(x) \check{f}(\xi, y)\, d\xi \tag{12}$$

and

$$\hat{f}(\eta, \xi) = \int_A \eta(y) \check{f}(\xi, y)\, dy. \tag{13}$$

We now write

$$\begin{aligned}
\hat{\phi}_1(\xi, \eta)\hat{\phi}_3(\mu, \nu) &= \int_{A^4} \frac{\xi(y)\mu(v)}{\eta(x)\nu(u)} \phi_1(x, y)\phi_3(u, v)\, dx\, dy\, du\, dv = \\
&= \int_{A^5} \frac{\xi(y)\mu(v)}{\eta(x)\nu(u)} \phi_4(uy, v\bar{z})\phi_2(xyuv\bar{z}, z)\phi_0(xv, y\bar{z})\, dz\, dx\, dy\, du\, dv = \\
&= \int_{A^5} \frac{\xi(yz)\mu(vz)}{\eta(x\bar{v}\bar{z})\nu(u\bar{y}\bar{z})} \phi_4(u, v)\phi_2(xu\bar{z}, z)\phi_0(x, y)\, dz\, dx\, dy\, du\, dv,
\end{aligned} \tag{14}$$

where we change the variables $x \mapsto x\bar{v}$ and $u \mapsto u\bar{y}$ followed by $y \mapsto yz$ and $v \mapsto vz$ in the last equality. Applying (12) to ϕ_2, then collecting all characters applied to z, and applying (13), we continue

$$\begin{aligned}
\hat{\phi}_1(\xi, \eta)\hat{\phi}_3(\mu, \nu) &= \int_{\hat{A}\times A^5} \frac{\xi(yz)\mu(vz)\sigma(xu\bar{z})}{\eta(x\bar{v}\,\bar{z})\nu(u\bar{y}\,\bar{z})} \phi_4(u, v)\check{\phi}_2(\sigma, z)\phi_0(x, y)\, d\sigma\, dz\, dx\, dy\, du\, dv = \\
&= \int_{A^5\times\hat{A}} \frac{\xi(y)\mu(v)\sigma(xu)\xi\eta\mu\nu\bar{\sigma}(z)}{\eta(x\bar{v})\nu(u\bar{y})} \phi_4(u, v)\check{\phi}_2(\sigma, z)\phi_0(x, y)\, dz\, dx\, dy\, du\, dv\, d\sigma = \\
&= \int_{A^4\times\hat{A}} \frac{\xi\nu(y)\mu\eta(v)}{\eta\bar{\sigma}(x)\nu\bar{\sigma}(u)} \phi_4(u, v)\hat{\phi}_2(\xi\eta\mu\nu\bar{\sigma}, \sigma)\phi_0(x, y)\, dx\, dy\, du\, dv\, d\sigma,
\end{aligned}$$

where we collect all characters applied to x, y, u, and v in the last equality. Finally, applying (10) to ϕ_0 and ϕ_4, we obtain the integral

$$\hat{\phi}_1(\xi, \eta)\hat{\phi}_3(\mu, \nu) = \int_{\hat{A}} \hat{\phi}_4(\mu\eta, \nu\bar{\sigma})\hat{\phi}_2(\xi\eta\mu\nu\bar{\sigma}, \sigma)\hat{\phi}_0(\xi\nu, \eta\bar{\sigma})\, d\sigma.$$

3. Functions of the automorphic beta pentagon type

Let A be a locally compact Abelian group, $B \subset A$ be a subgroup, $g \in \widehat{B}^{[2]}$, and $h\colon A \to \widehat{B}$, $x \mapsto h_x$, be a group homomorphism such that the map $\varepsilon\colon B \to \mathbb{T}$, $b \mapsto h_b(b)$, is a character. We say that a function $\phi \in \mathbb{C}^{[4]\times A^2}$ is of the *automorphic beta pentagon type* (B, g, h) if it satisfies beta pentagon relation (2) and the automorphicity conditions

$$\phi_i(bx, y) = \gamma_i h_y(b)\phi_i(x, y), \quad (i, b, x, y) \in [4] \times B \times A^2, \tag{15}$$

where

$$\gamma_0 = g(0), \qquad \gamma_1 = g(0)g(1), \qquad \gamma_2 = g(1), \qquad \gamma_3 = g(1)g(2), \qquad \gamma_4 = g(2). \tag{16}$$

Our first main result is the following theorem.

Theorem 1. *Let $\phi \in \mathbb{C}^{[4]\times A^2}$ be of the automorphic beta pentagon type (B, g, h). Then for any characters $\alpha, \beta \in \widehat{B}$, the function $\psi \in \mathbb{C}^{[4]\times A^2}$ defined by*

$$\psi_i(x, y) = \int_B \phi_i(x, yb)\mu_i h_{bx}(b)\, db, \quad (i, x, y) \in [4] \times A^2, \tag{17}$$

where

$$\mu_0 = \alpha\gamma_3, \qquad \mu_1 = \alpha, \qquad \mu_2 = \alpha\beta\gamma_0\gamma_2\gamma_4, \qquad \mu_3 = \beta, \qquad \mu_4 = \beta\gamma_1, \tag{18}$$

satisfies the relations

$$\begin{aligned} \psi_i(bx, y) &= \gamma_i h_y(b)\psi_i(x, y), \\ \psi_i(x, by) &= \varepsilon\bar{\mu}_i h_{\bar{x}}(b)\psi_i(x, y), \end{aligned} \quad (i, b, x, y) \in [4] \times B \times A^2, \tag{19}$$

$$\psi_1(x, y)\psi_3(u, v) = \int_{A/B} \psi_4(uy, v\bar{z})\psi_2(xyuv\bar{z}, z)\psi_0(xv, y\bar{z})\, dz. \tag{20}$$

Before proving the theorem, we first prove some auxiliary statements.

Lemma 1. *For a given locally compact Abelian group A, subgroup $B \subset A$, and group homomorphism $h\colon A \to \widehat{B}$, $x \mapsto h_x$, the map $\varepsilon\colon B \to \mathbb{T}$, $b \mapsto h_b(b)$, is a group homomorphism if and only if*

$$h_b(c)h_c(b) = 1, \quad (b, c) \in B^2. \tag{21}$$

Proof. We have

$$\varepsilon(bc) = h_{bc}(bc) = h_{bc}(b)h_{bc}(c) = h_b(b)h_c(b)h_b(c)h_c(c) = \varepsilon(b)\varepsilon(c)h_c(b)h_b(c), \tag{22}$$

and hence

$$\varepsilon(bc) = \varepsilon(b)\varepsilon(c) \Leftrightarrow h_c(b)h_b(c) = 1. \tag{23}$$

Lemma 2. *For a given locally compact Abelian group A, subgroup $B \subset A$, character $\gamma \in \widehat{B}$, and group homomorphism $h\colon A \to \widehat{B}$, $x \mapsto h_x$, such that (21) is satisfied, let $f \in \mathbb{C}^{A^2}$ be such that*

$$f(bx, y) = \gamma h_y(b) f(x, y), \quad (b, x, y) \in B \times A^2. \tag{24}$$

Then the function $\tilde{f} \in \mathbb{C}^{A^2\times\widehat{B}}$ defined by

$$\tilde{f}(x, y, \xi) = \int_B \xi(b) f(x, by)\, db \tag{25}$$

satisfies the relations

$$\begin{aligned} \tilde{f}(bx, y, \xi) &= \gamma h_y(b)\tilde{f}(x, y, \xi h_{\bar{b}}), \\ \tilde{f}(x, by, \xi) &= \bar{\xi}(b)\tilde{f}(x, y, \xi), \end{aligned} \quad (b, x, y, \xi) \in B \times A^2 \times \widehat{B}. \tag{26}$$

Proof. We have

$$\tilde{f}(bx,y,\xi)=\int_B \xi(b')f(bx,b'y)\,db'=\int_B \xi(b')\gamma h_{b'y}(b)f(x,b'y)\,db'=$$
$$=\gamma h_y(b)\int_B \xi(b')h_{b'}(b)f(x,b'y)\,db'=\gamma h_y(b)\int_B \xi(b')h_{\bar{b}}(b')f(x,b'y)\,db'=$$
$$=\gamma h_y(b)\tilde{f}(x,y,\xi h_{\bar{b}}) \tag{27}$$

and

$$\tilde{f}(x,by,\xi)=\int_B \xi(b')f(x,b'by)\,db'=\int_B \xi(b'\bar{b})f(x,b'y)\,db'=$$
$$=\bar{\xi}(b)f(x,y,\xi). \tag{28}$$

Proof of Theorem 1. We start by noting that

$$\psi_i(x,y)=\omega_i(x,y,\mu_i), \tag{29}$$

where

$$\omega_i(x,y,\xi)\equiv\tilde{\phi}_i(x,y,\varepsilon h_x\xi),\quad (i,x,y,\xi)\in[4]\times A^2\times\widehat{B}. \tag{30}$$

Using Lemma 2, we can verify (19):

$$\psi_i(bx,y)=\omega_i(bx,y,\mu_i)=\tilde{\phi}_i(bx,y,\varepsilon h_{bx}\mu_i)=$$
$$=\gamma_i h_y(b)\tilde{\phi}_i(x,y,\varepsilon h_x\mu_i)=\gamma_i h_y(b)\omega_i(x,y,\mu_i)=\gamma_i h_y(b)\psi_i(x,y)$$

and

$$\psi_i(x,by)=\omega_i(x,by,\mu_i)=\tilde{\phi}_i(x,by,\varepsilon h_x\mu_i)=$$
$$=\varepsilon\bar{\mu}_i h_{\bar{x}}(b)\tilde{\phi}_i(x,y,\varepsilon h_x\mu_i)=\varepsilon\bar{\mu}_i h_{\bar{x}}(b)\omega_i(x,y,\mu_i)=\varepsilon\bar{\mu}_i h_{\bar{x}}(b)\psi_i(x,y).$$

To verify (20), we make the transformation

$$\omega_1(x,y,\xi\varepsilon h_{\bar{x}})\omega_3(u,v,\eta\varepsilon h_{\bar{u}})=\tilde{\phi}_1(x,y,\xi)\tilde{\phi}_3(u,v,\eta)=\int_{B^2}\xi(b)\eta(c)\phi_1(x,by)\phi_3(u,cv)\,db\,dc=$$
$$=\int_{A\times B^2}\xi(b)\eta(c)\phi_4(uby,cv\bar{z})\phi_2(xbyucv\bar{z},z)\phi_0(xcv,by\bar{z})\,dz\,db\,dc,$$

where we use the beta pentagon relation for ϕ in the last equality. We continue by applying (15) to ϕ_4 and ϕ_0:

$$\omega_1(x,y,\xi\varepsilon h_{\bar{x}})\omega_3(u,v,\eta\varepsilon h_{\bar{u}})=$$
$$=\int_{A\times B^2}\xi(b)\eta(c)\gamma_4 h_{cv\bar{z}}(b)\gamma_0 h_{by\bar{z}}(c)\phi_4(uy,cv\bar{z})\phi_2(bcxyuv\bar{z},z)\phi_0(xv,by\bar{z})\,dz\,db\,dc=$$
$$=\int_{A\times B^2}\xi\gamma_4 h_{v\bar{z}}(b)\eta\gamma_0 h_{y\bar{z}}(c)\phi_4(uy,cv\bar{z})\phi_2(bcxyuv\bar{z},z)\phi_0(xv,by\bar{z})\,dz\,db\,dc,$$

where we use (21). We continue by applying (15) to ϕ_2:

$$\omega_1(x,y,\xi\varepsilon h_{\bar{x}})\omega_3(u,v,\eta\varepsilon h_{\bar{u}}) =$$

$$= \int_{A\times B^2} \xi\gamma_4 h_{v\bar{z}}(b)\eta\gamma_0 h_{y\bar{z}}(c)\gamma_2 h_z(bc)\phi_4(uy,cv\bar{z})\phi_2(xyuv\bar{z},z)\phi_0(xv,by\bar{z})\,dz\,db\,dc =$$

$$= \int_{A\times B^2} \xi\gamma_4\gamma_2 h_v(b)\eta\gamma_0\gamma_2 h_y(c)\phi_4(uy,cv\bar{z})\phi_2(xyuv\bar{z},z)\phi_0(xv,by\bar{z})\,dz\,db\,dc =$$

$$= \int_{A} \tilde{\phi}_4(uy,v\bar{z},\eta\gamma_1 h_y)\phi_2(xyuv\bar{z},z)\tilde{\phi}_0(xv,y\bar{z},\xi\gamma_3 h_v)\,dz,$$

where we use (16) and transformation (25) in the last equality. We continue by splitting the integral over A into a double integral over B and A/B:

$$\omega_1(x,y,\xi\varepsilon h_{\bar{x}})\omega_3(u,v,\eta\varepsilon h_{\bar{u}}) =$$

$$= \int_{B\times A/B} \tilde{\phi}_4(uy,v\bar{z}\bar{b},\eta\gamma_1 h_y)\phi_2(xyuv\bar{z}\bar{b},bz)\tilde{\phi}_0(xv,y\bar{z}\bar{b},\xi\gamma_3 h_v)\,db\,dz =$$

$$= \int_{B\times A/B} \xi\eta\gamma_1\bar{\gamma}_2\gamma_3\varepsilon h_{yv\bar{z}}(b)\tilde{\phi}_4(uy,v\bar{z},\eta\gamma_1 h_y)\phi_2(xyuv\bar{z},bz)\tilde{\phi}_0(xv,y\bar{z},\xi\gamma_3 h_v)\,db\,dz,$$

where we use Lemma 2 and (15). We finish the calculation by absorbing the integral over B using definition (25):

$$\omega_1(x,y,\xi\varepsilon h_{\bar{x}})\omega_3(u,v,\eta\varepsilon h_{\bar{u}}) =$$

$$= \int_{A/B} \tilde{\phi}_4(uy,v\bar{z},\eta\gamma_1 h_y)\tilde{\phi}_2(xyuv\bar{z},z,\xi\eta\gamma_1\bar{\gamma}_2\gamma_3\varepsilon h_{yv\bar{z}})\tilde{\phi}_0(xv,y\bar{z},\xi\gamma_3 h_v)\,dz =$$

$$= \int_{A/B} \omega_4(uy,v\bar{z},\varepsilon\eta\gamma_1 h_{\bar{u}})\omega_2(xyuv\bar{z},z,\xi\eta\gamma_1\bar{\gamma}_2\gamma_3 h_{\bar{x}\bar{u}})\omega_0(xv,y\bar{z},\varepsilon\xi\gamma_3 h_{\bar{x}})\,dz.$$

Substituting $\xi = \alpha\varepsilon h_x$ and $\eta = \beta\varepsilon h_u$ in the obtained equality yields

$$\omega_1(x,y,\alpha)\omega_3(u,v,\beta) = \int_{A/B} \omega_4(uy,v\bar{z},\beta\gamma_1)\omega_2(xyuv\bar{z},z,\alpha\beta\gamma_1\bar{\gamma}_2\gamma_3)\omega_0(xv,y\bar{z},\alpha\gamma_3)\,dz, \tag{31}$$

which with (18) and (29) taken into account is equivalent to (20).

4. Functions of the Faddeev type

A function

$$f\colon [4]\times\mathbb{R}\to\mathbb{C},\quad (i,x)\mapsto f_i(x), \tag{32}$$

is said to be of the *Faddeev type* if it satisfies the operator relation

$$f_1(\mathsf{p})f_3(\mathsf{q}) = f_4(\mathsf{q})f_2(\mathsf{p}+\mathsf{q})f_0(\mathsf{p}), \tag{33}$$

where p and q are self-adjoint operators in a Hilbert space satisfying Heisenberg's commutation relation

$$\mathsf{pq}-\mathsf{qp} = (2\pi\mathsf{i})^{-1}. \tag{34}$$

Lemma 3. *A square-integrable function f is of the Faddeev type if and only if*

$$\tilde{f}_1(x)\tilde{f}_3(y) = e^{-2\pi \mathrm{i} xy} \int_{\mathbb{R}} \tilde{f}_4(y-z)\tilde{f}_2(z)\tilde{f}_0(x-z)e^{\pi \mathrm{i} z^2}\, dz \quad \forall\,(x,y) \in \mathbb{R}^2, \tag{35}$$

where

$$\tilde{f}(x) \equiv \int_{\mathbb{R}} e^{-2\pi \mathrm{i} xy} f(y)\, dy. \tag{36}$$

Proof. Using the inverse Fourier transform, we convert equality (33) into an operator-valued integral equality

$$\int_{\mathbb{R}^2} \tilde{f}_1(x)\tilde{f}_3(y)e^{2\pi \mathrm{i} x\mathsf{p}}e^{2\pi \mathrm{i} y\mathsf{q}}\, dx\, dy = \int_{\mathbb{R}^3} \tilde{f}_4(y)\tilde{f}_2(z)\tilde{f}_0(x)e^{2\pi \mathrm{i} y\mathsf{q}}e^{2\pi \mathrm{i} z(\mathsf{p}+\mathsf{q})}e^{2\pi \mathrm{i} x\mathsf{p}} dx\, dy\, dz, \tag{37}$$

which with the operator equalities

$$e^{2\pi \mathrm{i} x\mathsf{p}}e^{2\pi \mathrm{i} y\mathsf{q}} = e^{(2\pi \mathrm{i})^2 xy[\mathsf{p},\mathsf{q}]}e^{2\pi \mathrm{i} y\mathsf{q}}e^{2\pi \mathrm{i} x\mathsf{p}} = e^{2\pi \mathrm{i} xy}e^{2\pi \mathrm{i} y\mathsf{q}}e^{2\pi \mathrm{i} x\mathsf{p}} \tag{38}$$

and

$$e^{2\pi \mathrm{i} z(\mathsf{p}+\mathsf{q})} = e^{(2\pi \mathrm{i})^2 z^2[\mathsf{p},\mathsf{q}]/2}e^{2\pi \mathrm{i} z\mathsf{q}}e^{2\pi \mathrm{i} z\mathsf{p}} = e^{\pi \mathrm{i} z^2}e^{2\pi \mathrm{i} z\mathsf{q}}e^{2\pi \mathrm{i} z\mathsf{p}} \tag{39}$$

taken into account becomes

$$\begin{aligned} &\int_{\mathbb{R}^2} \tilde{f}_1(x)\tilde{f}_3(y)e^{2\pi \mathrm{i} xy}e^{2\pi \mathrm{i} y\mathsf{q}}e^{2\pi \mathrm{i} x\mathsf{p}}\, dx\, dy = \\ &\qquad = \int_{\mathbb{R}^3} \tilde{f}_4(y)\tilde{f}_2(z)\tilde{f}_0(x)e^{\pi \mathrm{i} z^2}e^{2\pi \mathrm{i}(y+z)\mathsf{q}}e^{2\pi \mathrm{i}(x+z)\mathsf{p}}\, dx\, dy\, dz = \\ &\qquad = \int_{\mathbb{R}^3} \tilde{f}_4(y-z)\tilde{f}_2(z)\tilde{f}_0(x-z)e^{\pi \mathrm{i} z^2}e^{2\pi \mathrm{i} y\mathsf{q}}e^{2\pi \mathrm{i} x\mathsf{p}}\, dx\, dy\, dz. \end{aligned} \tag{40}$$

Comparing the coefficients of the operators $e^{2\pi \mathrm{i} y\mathsf{q}}e^{2\pi \mathrm{i} x\mathsf{p}}$, we conclude that equality (40) holds if and only if equality (35) holds.

Taking the complex conjugate of (35), we also have

$$\bar{\tilde{f}}_1(x)\bar{\tilde{f}}_3(y) = e^{2\pi \mathrm{i} xy} \int_{\mathbb{R}} \bar{\tilde{f}}_4(y-z)\bar{\tilde{f}}_2(z)\bar{\tilde{f}}_0(x-z)e^{-\pi \mathrm{i} z^2}\, dz, \quad (x,y) \in \mathbb{R}^2. \tag{41}$$

Remark 1. If $f_i(x)$ is a function of the Faddeev type, then the function $g_i(x) = f_i(-x)$ is also of the Faddeev type.

Example 1. The constant unit function

$$f_i(x) = 1, \quad i \in [4], \tag{42}$$

is trivially of the Faddeev type.

Example 2. The Gaussian exponentials

$$f_j(x) = a_j e^{-b_j x^2}, \quad j \in [4], \quad a \in \mathbb{R}^{[4]}, \quad b \in \mathbb{R}^{[4]}_{>0}, \tag{43}$$

form a function of the Faddeev type with an appropriate choice of the constants a_j and b_j.

Example 3. A nontrivial and interesting example of a function of the Faddeev type is given by Faddeev's quantum dilogarithm [7]:

$$f_j(x) = \Phi_\hbar(x) \equiv \exp\left(\int_{\mathbb{R}+\mathrm{i}\epsilon} \frac{e^{-\mathrm{i}2xz}}{4\sinh(zb)\sinh(zb^{-1})}\frac{dz}{z}\right), \quad j \in [4], \tag{44}$$

where $\hbar \in \mathbb{R}_{>0}$, b is any root of the equation

$$(b + b^{-1})^{-2} = \hbar, \tag{45}$$

and ϵ is an arbitrary sufficiently small positive real number. It is convenient to choose the unique b such that $0 \le \arg b < \pi/2$ and $\sqrt{\hbar} = (b + b^{-1})^{-1} > 0$.

The integral in (44) converges absolutely and admits an analytic continuation to complex x with $|\operatorname{Im} x| < 1/2\sqrt{\hbar}$. In the case where $\arg b > 0$ (i.e., $\hbar > 1/4$), it can be shown that

$$\Phi_\hbar(x) = \frac{(-qe^{2\pi bx}; q^2)_\infty}{(-\bar{q}e^{2\pi b^{-1}x}; \bar{q}^2)_\infty}, \qquad q := e^{\pi\mathrm{i}b^2}, \quad \bar{q} := e^{-\pi\mathrm{i}b^{-2}}, \tag{46}$$

where we use the standard notation of the theory of basic hypergeometric series

$$(x; q)_\infty \equiv \prod_{n=0}^{\infty}(1 - xq^n), \quad |q| < 1. \tag{47}$$

Equation (46) can be used to analytically continue the definition of $\Phi_\hbar(x)$ to the entire complex plane. It is straightforward to see that it satisfies the functional equations

$$\Phi_\hbar\left(x - \frac{\mathrm{i}b^{\pm 1}}{2}\right) = (1 + e^{2\pi b^{\pm 1}x})\Phi_\hbar\left(x + \frac{\mathrm{i}b^{\pm 1}}{2}\right). \tag{48}$$

We also have the inversion relation

$$\Phi_\hbar(x)\Phi_\hbar(-x) = e^{\pi\mathrm{i}x^2}e^{-\pi\mathrm{i}(2+\hbar^{-1})/12}. \tag{49}$$

Faddeev's quantum dilogarithm is closely related to Shintani's double sine function [8]–[10], but the pentagon identity (33) seems to have been unknown before Faddeev's paper [11] (see, e.g., [12]–[15] for further properties of Faddeev's quantum dilogarithm).

Our second main result is the following theorem.

Theorem 2. *Let f and g be two functions of the Faddeev type. Then the function*

$$\begin{aligned}&\varphi\colon [4] \times \mathbb{R}^2 \to \mathbb{C},\\ &(j, x, y) \mapsto \varphi_j(x, y) = \int_{\mathbb{R}} e^{2\pi\mathrm{i}yt} f_j\left(t + \frac{x}{2}\right)\bar{g}_j\left(t - \frac{x}{2}\right)dt,\end{aligned} \tag{50}$$

is of the beta pentagon type over $\mathbb{R}$, i.e., it satisfies relation (2) *with $A = \mathbb{R}$.*

It is instructive to give an operator interpretation for formula (50). If we define the Fourier operator F by

$$(\mathsf{F}f)(x) = \int_{\mathbb{R}} e^{2\pi \mathrm{i} xy} f(y)\, dy, \tag{51}$$

which is unitary in $L^2(\mathbb{R})$ because of the Fourier inversion formula, we have the relations

$$\mathsf{pF} = \mathsf{Fq}, \qquad \mathsf{qF} = -\mathsf{Fp}, \tag{52}$$

where

$$\mathsf{p}f(x) = \frac{1}{2\pi \mathrm{i}} \frac{\partial f(x)}{\partial x}, \qquad \mathsf{q}f(x) = xf(x). \tag{53}$$

Using Dirac's bra–ket notation for the scalar product in $L^2(\mathbb{R})$,

$$\langle f|g\rangle \equiv \int_{\mathbb{R}} \bar{f}(x) g(x)\, dx \tag{54}$$

for any $(x, y) \in \mathbb{R}^2$, we obtain

$$\begin{aligned}
\varphi_j(x,y) &\equiv \int_{\mathbb{R}} e^{2\pi \mathrm{i} yt} f_j\left(t + \frac{x}{2}\right) \bar{g}_j\left(t - \frac{x}{2}\right) dt = \\
&= \int_{\mathbb{R}} \overline{(e^{-\pi \mathrm{i} x \mathsf{p}} g_j)}(t) e^{2\pi \mathrm{i} yt} (e^{\pi \mathrm{i} x \mathsf{p}} f_j)(t)\, dt = \\
&= \int_{\mathbb{R}} \overline{(e^{-\pi \mathrm{i} x \mathsf{p}} g_j)}(t) (e^{2\pi \mathrm{i} y \mathsf{q}} e^{\pi \mathrm{i} x \mathsf{p}} f_j)(t)\, dt = \langle e^{-\pi \mathrm{i} x \mathsf{p}} g_j | e^{2\pi \mathrm{i} y \mathsf{q}} e^{\pi \mathrm{i} x \mathsf{p}} f_j \rangle = \\
&= \langle g_j | e^{\pi \mathrm{i} x \mathsf{p}} e^{2\pi \mathrm{i} y \mathsf{q}} e^{\pi \mathrm{i} x \mathsf{p}} f_j \rangle = \langle g_j | e^{2\pi \mathrm{i}(x\mathsf{p} + y\mathsf{q})} f_j \rangle.
\end{aligned} \tag{55}$$

This formula allows easily proving the following lemma.

Lemma 4. *Let $\varphi_j(x, y)$ be defined as in* (50). *Then we have the equality*

$$\varphi_j(x,y) = \int_{\mathbb{R}} e^{2\pi \mathrm{i} xt} \tilde{f}_j\left(t - \frac{y}{2}\right) \bar{\tilde{g}}_j\left(t + \frac{y}{2}\right) dt, \tag{56}$$

where

$$\tilde{f} \equiv \mathsf{F}^{-1} f, \quad f \in L^2(\mathbb{R}). \tag{57}$$

Proof. Indeed, using (55) and (52), we obtain

$$\begin{aligned}
\varphi_j(x,y) &= \langle g_j | e^{2\pi \mathrm{i}(x\mathsf{p} + y\mathsf{q})} f_j \rangle = \langle g_j | e^{2\pi \mathrm{i}(x\mathsf{p} + y\mathsf{q})} \mathsf{F}\mathsf{F}^{-1} f_j \rangle = \\
&= \langle g_j | \mathsf{F} e^{2\pi \mathrm{i}(x\mathsf{q} - y\mathsf{p})} \mathsf{F}^{-1} f_j \rangle = \langle \mathsf{F}^{-1} g_j | e^{2\pi \mathrm{i}(x\mathsf{q} - y\mathsf{p})} \mathsf{F}^{-1} f_j \rangle \equiv \\
&\equiv \langle \tilde{g}_j | e^{2\pi \mathrm{i}(x\mathsf{q} - y\mathsf{p})} \tilde{f}_j \rangle,
\end{aligned} \tag{58}$$

and formula (56) now follows by applying (55) backwards with f_j and g_j replaced with $\tilde{f}_j$ and $\tilde{g}_j$.

Proof of Theorem 2. Using (56), we write

$$\varphi_1(x,y)\varphi_3(u,v)=\int_{\mathbb{R}^2} e^{2\pi\mathsf{i}(xs+ut)}\tilde{f}_1\left(s-\frac{y}{2}\right)\tilde{f}_3\left(t-\frac{v}{2}\right)\bar{\tilde{g}}_1\left(s+\frac{y}{2}\right)\bar{\tilde{g}}_3\left(t+\frac{v}{2}\right)ds\,dt.$$

We then continue by applying (35) and (41)

$$\varphi_1(x,y)\varphi_3(u,v)=\int_{\mathbb{R}^4} e^{2\pi\mathsf{i}(xs+ut+vs+yt)+\pi\mathsf{i}(z^2-w^2)}\tilde{f}_4\left(t-\frac{v}{2}-z\right)\bar{\tilde{g}}_4\left(t+\frac{v}{2}-w\right)\times$$
$$\times\tilde{f}_2(z)\bar{\tilde{g}}_2(w)\tilde{f}_0\left(s-\frac{y}{2}-z\right)\bar{\tilde{g}}_0\left(s+\frac{y}{2}-w\right)ds\,dt\,dz\,dw.$$

Changing the integration variables $s\mapsto s+(z+w)/2$ and $t\mapsto t+(z+w)/2$, we can integrate over s and t using (56),

$$\varphi_1(x,y)\varphi_3(u,v)=$$
$$=\int_{\mathbb{R}^2} e^{\pi\mathsf{i}(x+y+u+v+z-w)(z+w)}\varphi_4(y+u,v+z-w)\tilde{f}_2(z)\bar{\tilde{g}}_2(w)\varphi_0(x+v,y+z-w)\,dz\,dw=$$
$$=\int_{\mathbb{R}^2} e^{\pi\mathsf{i}(x+y+u+v+z)(z+2w)}\varphi_4(y+u,v+z)\tilde{f}_2(z+w)\bar{\tilde{g}}_2(w)\varphi_0(x+v,y+z)\,dz\,dw,$$

where we shift $z\mapsto z+w$. Finally, shifting $w\mapsto w-z/2$ and again using (56), we obtain

$$\varphi_1(x,y)\varphi_3(u,v)=$$
$$=\int_{\mathbb{R}^2} e^{2\pi\mathsf{i}(x+y+u+v+z)w}\varphi_4(y+u,v+z)\tilde{f}_2\left(w+\frac{z}{2}\right)\bar{\tilde{g}}_2\left(w-\frac{z}{2}\right)\varphi_0(x+v,y+z)\,dz\,dw=$$
$$=\int_{\mathbb{R}}\varphi_4(y+u,v+z)\varphi_2(x+y+u+v+z,-z)\varphi_0(x+v,y+z)\,dz.$$

Example 4. If we take $f_j(x)=g_j(x)=\Phi_\hbar(x)$, then we have

$$\varphi_j(x,y)=\varphi^+(x,y)\equiv\int_{\mathbb{R}}\frac{\Phi_\hbar(t+x/2)}{\Phi_\hbar(t-x/2)}e^{2\pi\mathsf{i}ty}\,dt=$$
$$=\Psi_\hbar\left(x-\frac{\mathsf{i}}{2\sqrt{\hbar}}\right)\Psi_\hbar\left(y+\frac{\mathsf{i}}{2\sqrt{\hbar}}\right)\Psi_\hbar\left(-x-y+\frac{\mathsf{i}}{2\sqrt{\hbar}}\right),\tag{59}$$

where

$$\Psi_\hbar(x)\equiv\frac{\Phi_\hbar(x)}{\Phi_\hbar(0)}e^{-\pi\mathsf{i}x^2/2}.\tag{60}$$

The corresponding beta pentagon identity was first obtained in [3] as a limit case of Spiridonov's elliptic beta integral [16].

Example 5. If we take $f_j(x)=g_j(-x)=\Phi_\hbar(x)$, then we have

$$\varphi_j(x,y)=\varphi^-(x,y)\equiv\int_{\mathbb{R}}\frac{\Phi_\hbar(x/2+t)}{\Phi_\hbar(x/2-t)}e^{2\pi\mathsf{i}ty}\,dt,\tag{61}$$

and, unlike in Example 4, it is unknown, at least to us, if the integral can be simplified any further. It is interesting that the function $\varphi^-(x,y)$ is real valued, and it is related to Example 4 by the Fourier transform (cf. (10)). Namely, we have

$$2\varphi^-(-2x,-2y)=\int_{\mathbb{R}^2} e^{2\pi\mathsf{i}(xv-yu)}\varphi^+(u,v)\,du\,dv.\tag{62}$$

Example 6. If we take $f_j(x) = \Phi_\hbar(x)$ and $g_j(x) = 1$, then we have

$$\varphi_j(x, y) = e^{-\pi\mathsf{i}xy}(\mathsf{F}\Phi_\hbar)(y) = e^{-\pi\mathsf{i}(x+y)y}\Phi_\hbar\left(y + \frac{\mathsf{i}}{2\sqrt{\hbar}}\right)e^{\pi\mathsf{i}(1+1/\hbar)/12}. \tag{63}$$

This example has a specific quasiperiodicity property

$$\varphi_j(x + 1, y) = \varphi_j(x, y)e^{-\pi\mathsf{i}y}, \tag{64}$$

which allows applying Theorem 1, where $A = \mathbb{R}$, $B = \mathbb{Z}$, and $g(i) = 1$ for any $i \in [2]$. Hence, $\gamma_i = 1$, and $h_x(m) = e^{-\pi\mathsf{i}mx}$, and the associated character is therefore given by $\varepsilon(m) = h_m(m) = (-1)^m$. Choosing $\alpha = \beta = 1$, we also have $\mu_i = 1$, and the corresponding function $\psi \in \mathbb{C}^{[4]\times\mathbb{R}^2}$ becomes

$$\begin{aligned}\psi_j(x, y) &= \sum_{m\in\mathbb{Z}} \varphi_j(x, y + m)e^{-\pi\mathsf{i}(x+m)m} = \\ &= e^{-\pi\mathsf{i}xy} \sum_{m\in\mathbb{Z}} (\mathsf{F}\Phi_\hbar)(y + m)e^{-2\pi\mathsf{i}(x+1/2)m} \equiv \psi(x, y),\end{aligned} \tag{65}$$

which has the quasiperiodicity properties

$$\begin{aligned}\psi(x + m, y) &= h_y(m)\psi(x, y) = e^{-\pi\mathsf{i}my}\psi(x, y),\\ \psi(x, y + m) &= \varepsilon(m)h_x(-m)\psi(x, y) = (-1)^m e^{\pi\mathsf{i}mx}\psi(x, y)\end{aligned} \tag{66}$$

and the five-term integral relation

$$\psi(x, y)\psi(u, v) = \int_0^1 \psi(u + y, v - z)\psi(x + y + u + v - z, z)\psi(x + v, y - z)\, dz. \tag{67}$$

We note that the integrand is a periodic function of z, as it should be according to the general theory (see Sec. 3). It was shown in [4] that $\psi(x, y)$ admits an analytic continuation to $\mathbb{C}^2$ as a meromorphic function, and it has been used to reformulate the Teichmüller TQFT.

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